EE120 - Fall'19 - Lecture 16 Notes[1]

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Transfer Functions of LTI Systems

Section 9.7 *in* Oppenheim & Willsky

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

From the convolution property:

$$Y(s) = H(s)X(s)$$

where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$ is called the "transfer function" or "system function."

Transfer function from differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Take the Laplace transform of both sides and use differentiation property:

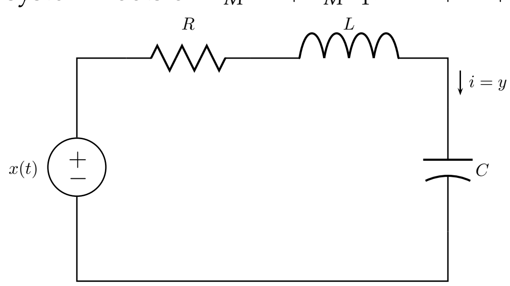
$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_M s^M + b_{M-1}s^{M-1} + ... + b_0}{a_N s^N + a_{N-1}s^{N-1} + ... + a_0}}$$

Poles of the system: roots of $a_N s^N + a_{N-1}s^{N-1} + ... + a_0$

Zeros of the system: roots of $b_M s^M + b_{M-1}s^{M-1} + ... + b_0$

<ins>Example:</ins>



Consider the RLC circuit above and define the output to be the current: $y(t) := i(t)$. Then,

$$C\frac{d}{dt}\underbrace{\left(x(t) - Ry(t) - L\frac{dy}{dt}\right)}_{\text{voltage across capacitor}} = y(t)$$

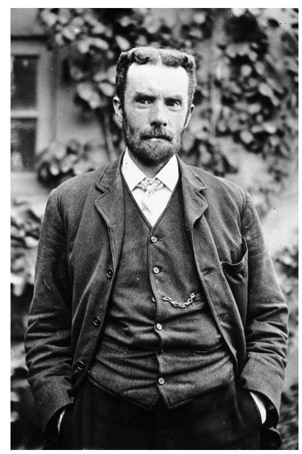


Figure 1: **Oliver Heaviside (1850-1925)**, a self-taught electrical engineer, invented the "operational calculus" where the differential operator $\frac{d}{dt}$ is treated as a symbol ('s' in our case) and a linear differential equation is manipulated algebraically. Dynamic circuit elements could now be represented with simple algebraic expressions similar to Ohm's Law (*e.g.* Ls for inductance). Heaviside's method had found widespread use by the time others established the full mathematical justification with the help of a transform used by Laplace a century earlier. Heaviside had many other contributions, including condensing Maxwell's theory of electromagnetism into the four vector equations known today. He further coined the terms "inductance" and "impedance," and the unit step $u(t)$ is sometimes referred to as the Heaviside function.

Rearrange terms:

$$LC\frac{d^2y}{dt^2} + RC\frac{dy}{dt} + y = C\frac{dx}{dt}.$$

Then, the transfer function is:

$$H(s) = \frac{Cs}{LCs^2 + RCs + 1}.$$

Poles: $s = \frac{-RC \mp \sqrt{R^2C^2 - 4LC}}{2LC}$. Zero: $s = 0$.

How do poles affect the system response?

If there are no repeated poles, partial fraction expansion gives:

$$H(s) = \sum_{i=1}^{N} \frac{A_i}{s - \alpha_i} \tag{1}$$

where α_i, $i = 1, ..., N$, are the poles. Then, assuming causality:

$$h(t) = \sum_{i=1}^{N} A_i e^{\alpha_i t} u(t) \tag{2}$$

Each pole α_i contributes an exponential term $e^{\alpha_i t}$ to the response.[2]

[2] See Figure 2 on the next page which we discussed in Lecture 3.

If α_i is repeated m times, then the system response includes:

$$t^{m-1}e^{\alpha_i t},\ ...,\ te^{\alpha_i t},\ e^{\alpha_i t}$$

<u>Example:</u> In the RLC circuit above, we expect oscillatory response if

$$R^2C^2 < 4LC.$$

How do zeros affect the system response?

Suppose $s = \beta$ is a zero of $H(s)$, *i.e.*, $H(\beta) = 0$. Then:

$$e^{\beta t} \rightarrow \boxed{h(t)} \rightarrow y(t) = H(\beta)e^{\beta t} = 0$$

Thus, the zero $s = \beta$ blocks inputs of the form $e^{\beta t}$ from appearing at the output.

<u>Example:</u> In the RLC circuit above the zero at $s = 0$ blocks constant inputs: when $x(t) = e^{0t} \equiv 1$, $y(t) \equiv 0$.

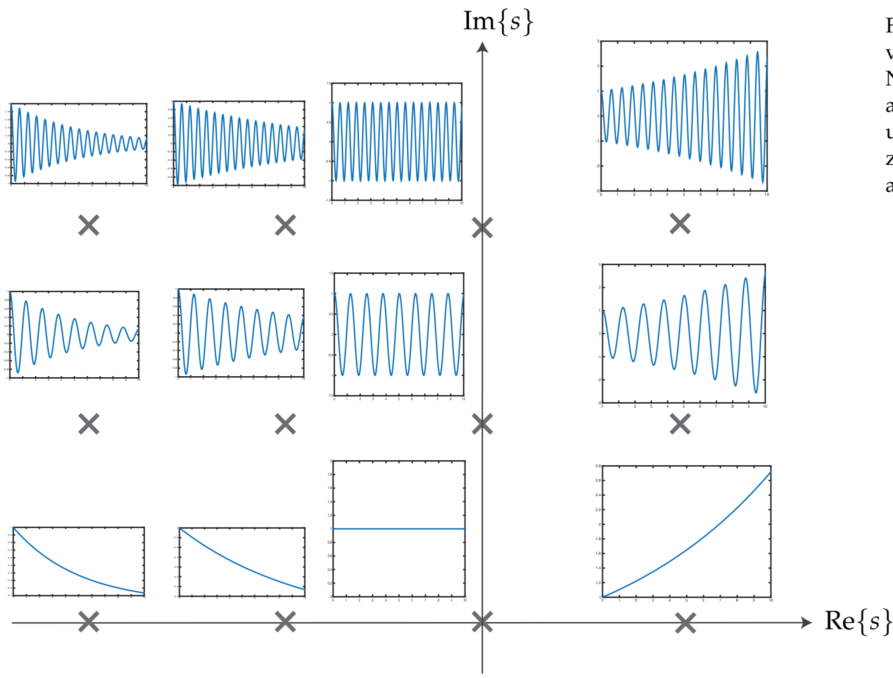


Figure 2: The real part of e^{st} for various values of s in the complex plane. Note that e^{st} is oscillatory when s has an imaginary component. It grows unbounded when $\mathrm{Re}\{s\} > 0$, decays to zero when $\mathrm{Re}\{s\} < 0$, and has constant amplitude when $\mathrm{Re}\{s\} = 0$.

Determining Stability from $H(s)$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \qquad\qquad Y(s) = H(s)X(s)$$

Causality: $h(t) = 0 \ \forall t < 0$ Stability: $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

> A causal LTI system with rational transfer function $H(s)$ is stable if and only if all poles of $H(s)$ have strictly negative real parts (i.e., they lie to the left of the imaginary axis).

The stability criterion above follows from two observations:

1) If $H(s)$ is rational, causality is equivalent to the ROC being the half plane to the right of the rightmost pole.

2) The absolute integrability condition $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ means the imaginary axis is within the ROC.

Example: A causal LTI system with transfer function $H(s) = \frac{1}{s+1}$ is stable, because the only pole is $s = -1$, which is negative. Indeed the impulse response is $h(t) = e^{-t}u(t)$ which is absolutely integrable. Note that $h(t) = -e^{-t}u(-t)$ gives the same $H(s)$ but is ruled out by causality.

Example:
$$H(s) = \frac{s-1}{(s+1)(s-2)} = \frac{2/3}{s+1} + \frac{1/3}{s-2}$$

is unstable because the pole $s = 2$ is positive. Due to causality the impulse response is:

$$h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}\underline{e^{2t}}\right)u(t)$$

which is not absolutely integrable.

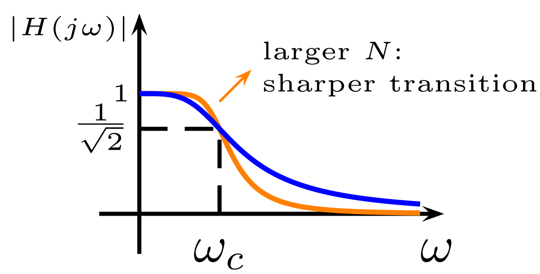
Example (poles on the imaginary axis cause instability):

$$H(s) = \frac{1}{s} \quad \text{(integrator)}$$

If the input is $x(t) = u(t)$, then $X(s) = \frac{1}{s}$ and $Y(s) = H(s)X(s) = \frac{1}{s^2}$. Then, $y(t) = tu(t)$ which is unbounded although $x(t)$ is bounded.

Example (Butterworth filters):

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}} \qquad \omega_c: \text{ cutoff frequency}, N: \text{ filter order}$$

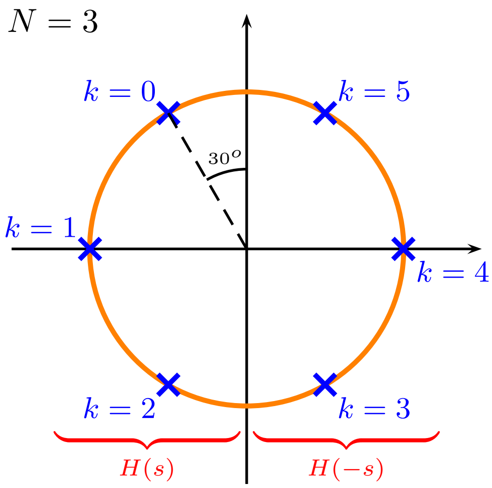


Derive the transfer function of a causal and stable LTI system with real-valued $h(t)$ that gives this frequency response.

$$|H(j\omega)|^2 = H(j\omega)\underbrace{H^*(j\omega)}_{\substack{H(-j\omega) \\ \text{since } h(t) \text{ is real}}} = \frac{1}{1 + \left(\frac{j\omega}{j\omega_c}\right)^{2N}} \implies H(s)H(-s) = \frac{1}{1 + \left(\frac{s}{j\omega_c}\right)^{2N}}$$

Thus, the roots of $\underbrace{1 + \left(\frac{s}{j\omega_c}\right)^{2N} = 0}$ are the poles of $H(s)$ combined with the poles of $H(-s)$.

$$\begin{aligned} \frac{s}{j\omega_c} &= e^{j\left(\frac{\pi}{2N} + k\frac{2\pi}{2N}\right)} \qquad k = 0, 1, ..., 2N-1 \\ s &= \underbrace{e^{j\frac{\pi}{2}}}_{j}\omega_c e^{j\left(\frac{\pi}{2N} + k\frac{2\pi}{2N}\right)} \end{aligned}$$



Since the filter is to be causal and stable, $H(s)$ must contain the N poles in the left-half plane ($k = 0, 1, ..., N-1$) and $H(-s)$ must contain the rest $k = N, ..., 2N-1$.

Denominator of $H(s)$ for $N = 3$:

$$(s+\omega_c)\underbrace{(s+\omega_c e^{j\frac{\pi}{3}})(s+\omega_c e^{-j\frac{\pi}{3}})}_{s^2+\underbrace{2cos(\frac{\pi}{3})}_{=1}\omega_c s+\omega_c^2}$$

$$= (s+\omega_c)(s^2+\omega_c s+\omega_c^2) = s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3$$

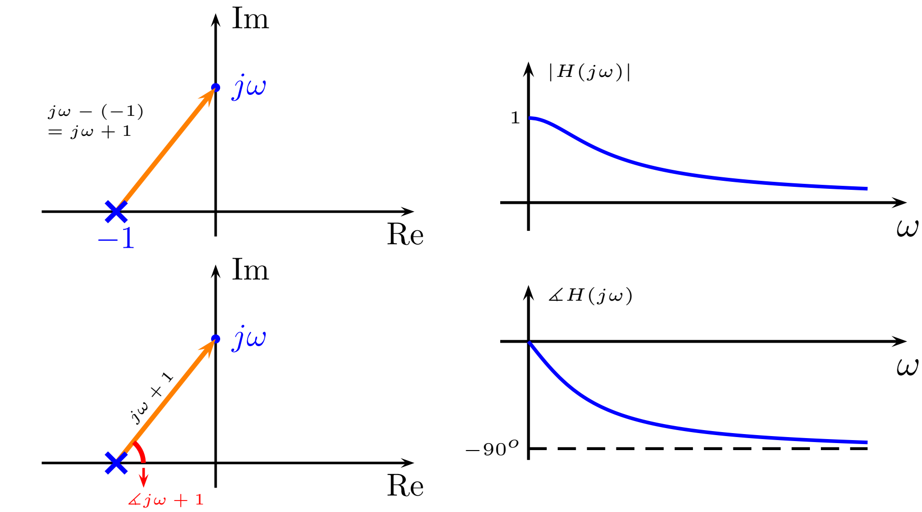
Therefore, $H(s) = \frac{\omega_c^3}{s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3}$ (so that $H(0) = $ dc-gain $= 1$)

Normalized transfer function for the $N = 3$ example above:

$$H^0(s) = \frac{1}{s^3+2s^2+2s+1} \qquad H(s) = H^0\left(\frac{s}{\omega_c}\right) \quad \text{for any desired } \omega_c$$

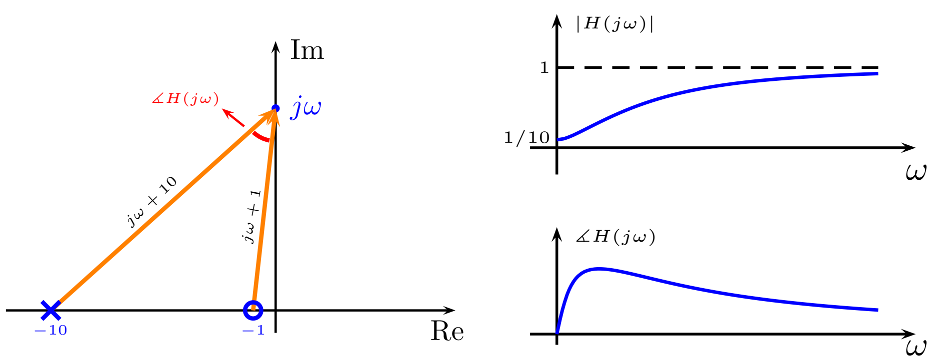
Evaluating the Frequency Response from the Pole-Zero Plot

Example: $H(s) = \frac{1}{s+1}$ $\qquad |H(j\omega)| = \frac{1}{|j\omega+1|}$



Note that $H(s) = \frac{1}{s+1}$ is a first order ($N = 1$) Butterworth filter. Try to apply this geometric interpretation of the frequency response to higher order Butterworth filters, such as $N = 3$ discussed in the previous example.

Example: To see the effect of a zero let $H(s) = \frac{s+1}{s+10}$.



Example (second order system):

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

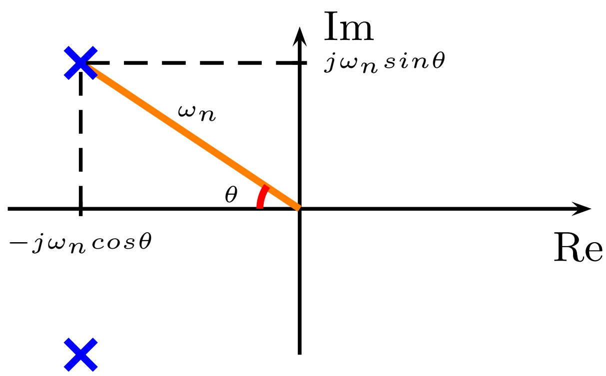
ζ : damping ratio, ω_n : natural frequency

Recall from Lecture 7: resonance occurs if $\zeta < \frac{1}{\sqrt{2}} \approx 0.7$

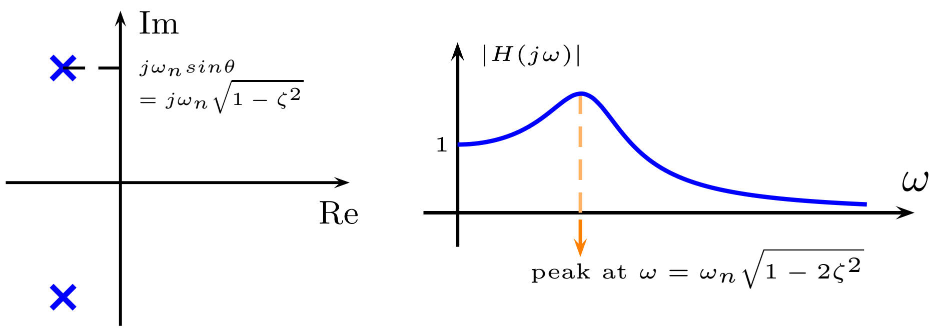
Poles of $H(s)$: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, or $\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1 = 0$.
Then, $\frac{s}{\omega_n} = -\zeta \mp \sqrt{\zeta^2 - 1}$

Therefore, complex conjugate poles if $\zeta < 1$:

$$s_{1,2} = \omega_n(-cos(\theta) \mp jsin(\theta)) \text{ where } \theta \text{ defined by } \boxed{cos\theta = \zeta}$$



Resonance condition $\zeta < \frac{1}{\sqrt{2}}$ means $\theta > 45^o$



See Figure 3 below which we discussed in Lecture 7.

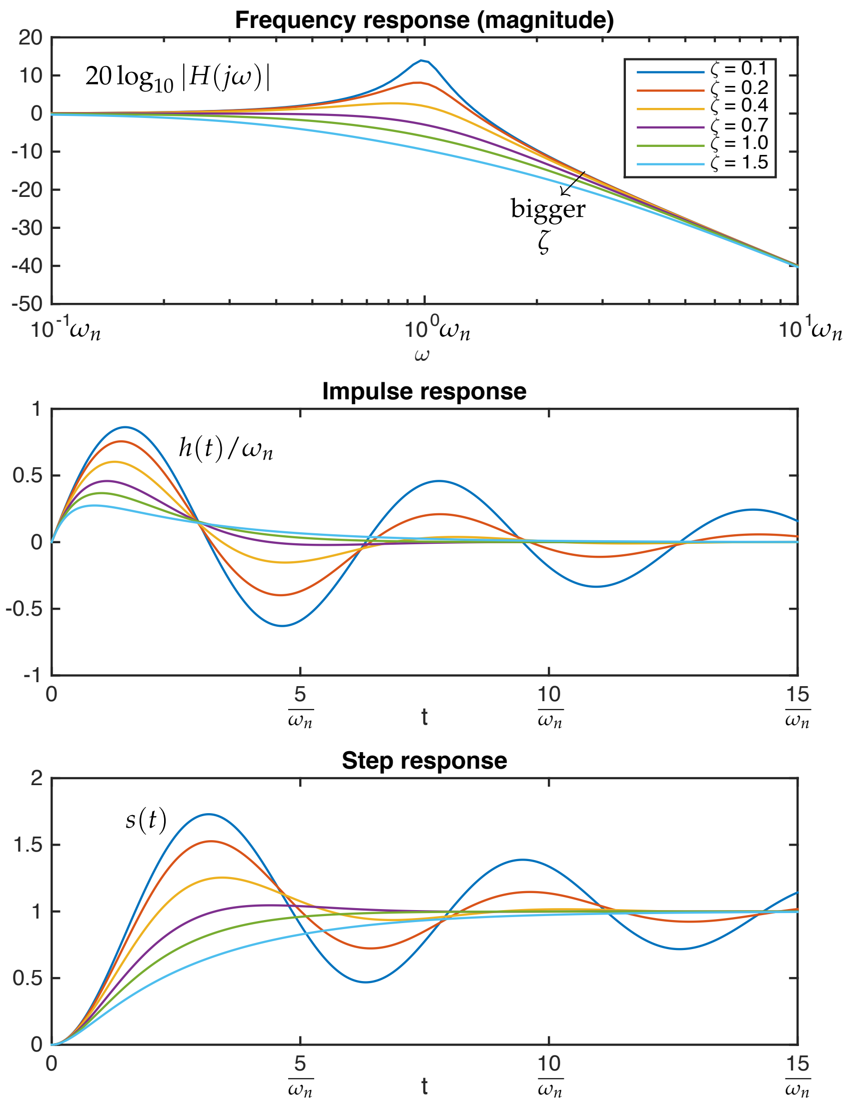


Figure 3: The frequency, impulse, and step responses for the second order system (3). Note from the frequency response (top) that a resonance peak occurs when $\zeta < 0.7$.